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Non-linear stability of convection in a porous medium with inclined temperature gradient

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Abstract—The energy method is used to discuss the non-linear stability of convection in a horizontal porous layer subjected to an inclined temperature gradient. The compound matrix method is used to solve the associated eigenvalue problem. It is noted that linear instability is superceded by subcritical finite amplitude instability. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The problem of the stability of the steady convective flow, which is caused by the horizontal component of the temperature gradient in a shallow horizontal layer of the porous medium, has been studied by Weber [1] and Nield [2, 3]. Both of these authors use linear stability analyses. Weber's analysis is concerned with a small non-dimensional horizontally applied temperature gradient, β; he used a perturbation theory in terms of the powers of β, to solve for the critical Rayleigh number. Nield's [2] analysis removed the restriction of small β and used a low order Galerkin approximation to solve the eigenvalue problem. In his subsequent paper Nield [3] noted that his earlier treatment of the problem, based upon the low order Galerkin approximation was not satisfactory, particularly when β increased considerably. He [3] then used a higher order Galerkin approximation and found improved results. In particular, he noted that the values of the critical vertical Rayleigh numbers R_V, rather than increasing indefinitely with the increase of the horizontal Rayleigh number R_H, now reached a maximum and then decreased to the value zero. Nield attributed this new result to predict that Hadley flow in a porous medium, when the circulation is sufficiently intense, becomes unstable even in the absence of an applied vertical gradient.

The purpose of the present paper is to study the non-linear stability of the title problem via an energy method. We use a compound matrix method to solve the eigenvalue problem and the golden section search method for determining the maximum and the minimum routines. We carry out both the linear and non-linear stability calculations using this method. Our calculations for the linear stability analysis compare well with Nield's [3] results, except with some minor changes in the values when the horizontal Rayleigh's number takes on higher values. The non-linear stability results are, however, new. We find that linear stability is superceded by subcritical finite amplitude instability. We also find that there exists a critical horizontal Rayleigh number (horizontal temperature gradient) at which the porous fluid layer is always unstable, irrespective of the vertical Rayleigh number.

We remark that linear stability analysis yields criteria sufficient only for instability and says nothing definite about stability. Likewise, the energy analysis predicts only the sufficient conditions for stability and says nothing definite about instability. The two types of analysis, thus, complement each other to some degree, and when the linearized systems of governing equations happen to be self-adjoint their predictions coincide. In other situations the energy analysis shows the possibility of subcritical instability which should be analysed over and above the linear instability analysis.

2. BASIC EQUATIONS

The model to be studied is the same as considered by Weber [1] and Nield [2, 3]; we follow the notation and scaling of Nield [3]. The porous medium occupies a layer of height H. The vertical temperature difference across the boundaries is ΔT and flow in the porous medium is governed by Darcy's law. For the density variation, the Boussinesq approximation is assumed to be valid. Accordingly, following the non-dimensionalization scheme of Nield [3], the governing equations take the form

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\mathbf{v} + \nabla P = T\mathbf{k} \tag{2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T = \nabla^2 T \tag{3}$$

where $\mathbf{v}$, P, and T are non-dimensionalized seepage

NOMENCLATURE

a	dimensionless overall horizontal wave number
D	differential operator, $\mathrm{d}/\mathrm{d}z$
g	gravitational acceleration
H	layer height
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the x-, y- and z-directions, respectively
K	permeability
l	dimensionless wave number in y-direction
P, p	dimensionless pressure/perturbed pressure
p_s	dimensionless steady state pressure
R_H	horizontal Rayleigh number
R_V	vertical Rayleigh number
T	dimensionless temperature
T_s	dimensionless steady-state temperature
t	dimensionless time
$\mathbf{u}, \mathbf{v}$	dimensionless perturbed velocity vector/velocity vector
$\mathbf{u}_s$	dimensionless steady-state velocity vector
x, y, z	dimensionless Cartesian coordinates.

Greek symbols

α_m	thermal diffusivity
β_T	horizontal temperature gradient
γ_T	coefficient of volume expansion
θ	perturbed dimensionless temperature
κ	thermal diffusivity
μ	dynamic viscosity
ρ_0	density at the reference temperature.

velocity, pressure and temperature, respectively, and $\mathbf{k}$ is the unit vector in the z-direction. For rigid boundaries, the nondimensional form of boundary conditions become

$$w = 0 \quad \text{and} \quad T = -(\pm R_V/2) - R_H x \quad \text{at} \quad z = \pm 1/2 \tag{4}$$

where R_V and R_H are vertical and horizontal Rayleigh numbers, respectively, and are defined as [3]

$$R_V = \rho_0 g \gamma_T K H \Delta T/\mu\alpha_m \quad R_H = \rho_0 g \gamma_T K H^2 \beta_T/\mu\alpha_m. \tag{5}$$

Here ρ_0 is the density at the reference temperature, g is the gravitational acceleration, γ_T is the thermal expansion coefficient, K is the permeability of the medium, β_T is the horizontal temperature gradient, μ is the dynamic viscosity, and α_m is the thermal diffusivity. The basic steady state solution ($\mathbf{u}_s, T_s, p_s$) of equations (1)–(3) satisfying the boundary condition (4) is [3]

$$u_s = R_H z \quad v_s = 0 \quad w_s = 0$$

$$T_s = -R_H x - R_V z + R_H^2 f(z)$$

$$\nabla p_s = T_s \mathbf{k} - \mathbf{u}_s \tag{6}$$

where we imposed the requirement that there is no net horizontal mass flux;

$$\int_{-1/2}^{1/2} u_s \, \mathrm{d}z = 0 \quad \int_{-1/2}^{1/2} v_s \, \mathrm{d}z = 0 \tag{7}$$

and where

$$f(z) = \tfrac{1}{24}(z - 4z^3). \tag{8}$$

3. STABILITY ANALYSIS

We now perturb the steady-state solution as follows:

$$\mathbf{v} = \mathbf{u}_s + \mathbf{u} \quad T = T_s + \theta \quad P = p_s + p. \tag{9}$$

The perturbation equations then take the form

$$\nabla \cdot \mathbf{u} = 0 \tag{10}$$

$$\mathbf{u} + \nabla p = \theta \mathbf{k} \tag{11}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla)\theta = \nabla^2 \theta - \mathbf{u}_s \cdot \nabla \theta - \mathbf{u} \cdot \nabla T_s \tag{12}$$

where $\mathbf{u}_s$ and T_s are given by equation (6). The corresponding boundary conditions become

$$w = 0 \quad \text{and} \quad \theta = 0 \quad \text{at} \quad z = \pm 1/2. \tag{13}$$

A quick look at the system of equations (10)–(12) tells us that the linearized system is not symmetric (self-adjoint) and, hence, the energy method will give different results from the linear stability method. We define an energy functional as

$$E(t) = \frac{\xi}{2} \|\theta\|^2 \tag{14}$$

where ξ is a positive coupling parameter. On multiplying equation (11) by $\mathbf{u}$, equation (12) by θ and integrating over V, we find (after using the boundary conditions and divergence theorem)

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\theta\|^2 = -\|\nabla \theta\|^2 - \langle (\mathbf{u} \cdot \nabla T_s)\theta \rangle \tag{15}$$

$$\|\mathbf{u}\|^2 = \langle \theta w \rangle. \tag{16}$$

Here V denotes a typical periodicity cell, $\langle\cdot\rangle$ denotes the integration over V, and $\|\cdot\|$ denotes the $L^2(V)$ norm. Equations (14)–(16) can be put in the form [4],

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mathscr{I} - \mathscr{D}, \tag{17}$$

where

$$\mathscr{I} = -\xi\langle(\mathbf{u}\cdot\nabla T_s)\theta\rangle + \langle\theta w\rangle \tag{18}$$

$$\mathscr{D} = \xi\|\nabla\theta\|^2 + \|\mathbf{u}\|^2. \tag{19}$$

We now define

$$m = \max_{\mathscr{H}} \frac{\mathscr{I}}{\mathscr{D}}, \tag{20}$$

where $\mathscr{H}$ is the space of admissible solutions. On combining equation (17) with equations (19)–(20) and by using Poincaré inequality, we can infer, for $0 < m < 1$, that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leqslant -2\pi^2(1-m)E. \tag{21}$$

Inequality (21) clearly indicates that for $0 < m < 1$, $E(t) \to 0$ at least exponentially as $t \to \infty$. Since $E(t)$ in equation (14) does not contain the term $\|\mathbf{u}\|^2$, the kinetic energy term for the velocity, it is worthwhile checking as to what happens to $\|\mathbf{u}\|^2$ as $t \to \infty$. Use of the Cauchy–Schwartz inequality on equation (16) implies that

$$\|\mathbf{u}\|^2 \leqslant \|\theta\|^2. \tag{22}$$

Thus equations (14) and (22) clearly indicate that decay of $E(t)$ ensures the decay of $\|\mathbf{u}\|^2$.

4. NUMERICAL RESULTS AND DISCUSSION

We now return to equation (20) and consider the maximum problem at the critical argument $m = 1$. The associated Euler–Lagrange equations become

$$-\xi\nabla T_s\cdot\mathbf{u} + w + 2\xi\nabla^2\theta = 0 \tag{23}$$

$$\xi\nabla T_s\theta - \theta\mathbf{k} + 2\mathbf{u} = \nabla\omega \tag{24}$$

where ω is a Lagrange multiplier introduced since $\mathbf{u}$ is solenoidal. On taking *curlcurl* of equation (24) and then taking the third component of resulting equation, we find

$$2\nabla^2 w - \nabla_1^2(g_1\theta) + \xi R_H \frac{\partial^2\theta}{\partial x\,\partial z} = 0 \tag{25}$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and

$$g_1 = 1 + \xi(R_V - R_H^2 f_1) \quad f_1 = \tfrac{1}{24}(1-12z^2). \tag{26}$$

Equations (23) and (24) can also be written as

$$\xi R_H u + g_1 w + 2\xi\nabla^2\theta = 0 \tag{27}$$

$$-\xi R_H\theta + 2u = \frac{\partial\omega}{\partial x}. \tag{28}$$

It has been shown by Nield [3] and also confirmed by Straughan and Walker [5], that the steady longitudinal mode is the most favourable mode of disturbances for this problem. We also checked it here and found it to agree with their prediction. From now on, we thus restrict ourselves to this situation only. We perform the standard normal mode analysis and look for the solution of the above, in the form

$$[u, w, \theta, \omega] = [u(z), w(z), \theta(z), \omega(z)]\exp[ily].$$

On eliminating the different variables we derive the corresponding eigenvalue problem, which can, after some rearrangement of terms, be written as

$$D^2 w = h_1 w + h_2\theta \tag{29}$$

$$D^2\theta = h_3 w + h_4\theta \tag{30}$$

where $D = \mathrm{d}/\mathrm{d}z$, $a^2 = l^2$. The variables $h_1, \ldots, h_4$, which are functions of z, are given as

$$h_1 = a^2, \quad h_2 = -\frac{a^2}{2}g_1$$

$$h_3 = -\frac{g_1}{2\xi}, \quad h_4 = a^2 - \frac{\xi}{4}R_H^2. \tag{31}$$

The relevant boundary conditions are

$$w = \theta = 0 \quad \text{at} \quad z = \pm 1/2. \tag{32}$$

We consider R_V as the eigenvalue with the remaining variables as parameters. The critical vertical Rayleigh number is defined by

$$R_E = \max_{\xi}\min_{a^2} R_V(R_H, a^2, \xi). \tag{33}$$

On letting $x_1 = w$, $x_2 = Dw$, $x_3 = \theta$, $x_4 = D\theta$; the system of equations (29)–(30) can be written in the matrix form as

$$\dot{\mathbf{X}} = \mathbf{AX}. \tag{34}$$

where $\mathbf{X} = (x_1, x_2, x_3, x_4)^{\perp}$ and $\mathbf{A}$ is a suitable coefficient matrix. The boundary conditions now take the form

$$x_1 = x_3 = 0, \quad \text{at} \quad z = \pm 1/2. \tag{35}$$

We next employ the compound matrix method and carry out the maximization and minimization routines by golden section search. Table 1 displays our computed results of linear and non-linear critical vertical Rayleigh number. For the purpose of comparison we have also included the Nield's [3] linear results obtained via the Galerkin approximation. The second and third columns represent Nield's results and these are denoted by R_{VN} and a_{LN}, respectively. The next two columns represent the linear stability results obtained by the compound matrix method. We note the results for R_V and a_L, nearly coincide with those obtained by Nield, up to the value of $R_H = 80$, but there is a general variation of these values for R_H greater than 90. In particular, we note that, when R_H

Table 1. Critical vertical Rayleigh number vs horizontal Rayleigh number

R_H	R_{VN}	a_{LN}	R_V	a_L	R_E	a_E
0.0000	39.48	3.14	39.4784	3.14159	39.4784	3.14159
10.000	42.01	3.14	42.0076	3.14185	40.7235	3.09208
20.000	49.56	3.15	49.5486	3.14575	44.2151	2.95465
30.000	62.28	3.16	61.9567	3.16340	49.3234	2.76244
40.000	79.24	3.20	78.9664	3.21522	55.2825	2.55843
50.000	100.9	3.28	100.117	3.34455	61.4103	2.37439
60.000	126.4	3.51	124.473	3.67219	67.1673	2.22451
70.000	154.0	4.22	149.186	4.67123	72.1108	2.11239
80.000	161.9	7.78	164.371	6.53124	75.8263	2.03908
90.000	143.5	7.73	160.999	7.73120	77.8499	2.00932
100.00	123.3	7.67	143.586	8.46277	76.1711	6.57897
110.00	101.4	7.61	115.387	9.24311	43.7370	7.69003
119.91	—	—	76.1049	10.0414	0.00000	8.74504
120.00	62.00	9.51	75.2456	9.99230	—	—
132.50	0.000	9.64	7.71396	10.8611	—	—
133.73	—	—	0.00000	10.9433	—	—

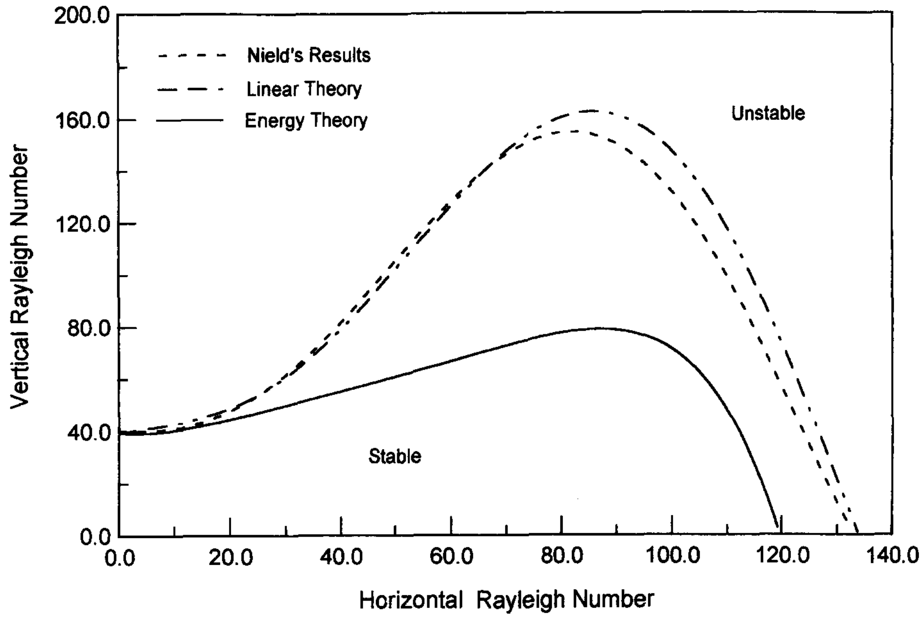

Fig. 1. Plots of critical values of vertical Rayleigh number R_V against the horizontal Rayleigh number R_H for various theories.

is increased over 90, values of R_{VN} fall faster than those of R_L, and a_L takes higher values than those of a_{LN}. Table 1 also gives the critical R_E and a_E values obtained via the energy method. A quick look clearly indicates that, overall, R_E and a_E values are smaller, compared to the linear case. This is not surprising, since, as is pointed out earlier, the energy method gives the sufficient condition for stability and, as such, the values obtained by this method are generally conservative as compared to the linear stability case. One significant different is found, in the a_E values, which unlike the linear case, take an abrupt jump when R_H is very close to 90. Figure 1 displays the graph of the vertical Rayleigh number vs horizontal Rayleigh number for the three cases. As it is clear from the graphs, there is the possibility of a subcritical instability, which needs to be further analysed by considering finite amplitude analysis.

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